Published by Institute of Physics Publishing for SISSA

Received: *June 3, 2008*
Accepted: *July 2, 2008*
Published: *July 17, 2008*

Wall crossing formula for $\mathcal{N} = 4$ dyons: a macroscopic derivation

Ashoke Sen

Harish-Chandra Research Institute,
Chhatnag Road, Jhusi, Allahabad 211019, India
E-mail: sen@mri.ernet.in, ashokesen1999@gmail.com

Abstract: We derive the wall crossing formula for the decay of a quarter BPS dyon into a pair of half-BPS dyons by analyzing the quantum dynamics of multi-centered black holes in $\mathcal{N} = 4$ supersymmetric string theories. Our analysis encompasses the cases where the final decay products are non-primitive dyons. The results are in agreement with the microscopic formula for the dyon spectrum in the special case of heterotic string theory on T^6.

Keywords: Black Holes in String Theory, Superstrings and Heterotic Strings.

JHEP07(2008)078

Much of the study of string theoretic black holes has focussed on BPS black holes since the degeneracy — or more precisely an appropriate index — associated with BPS states are protected and do not change under a continuous variation of the various moduli. Nevertheless it has been known for some time [1 – 3] that in some cases the index can jump across codimension one subspaces of the moduli space on which the original state becomes marginally unstable against possible decay into a pair of BPS states. Such subspaces are known as walls of marginal stability, and the jump in the index across these walls — known as the wall crossing formula — has been the subject of intense investigation in recent years [4 – 12]. While most of the work has been focussed on half-BPS dyonic black holes in $\mathcal{N} = 2$ supersymmetric string theories, by now we also have a good understanding of this phenomenon in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric string theories [13 – 20]. In fact in the latter theories the situation is somewhat better since we also have a good understanding of the exact spectrum of BPS dyons in these theories. This allows us to verify the general wall crossing formula derived from macroscopic considerations involving multi-centered black hole solutions [7] against explicit results obtained from microstate counting [16, 17].

For deriving the wall crossing formula for walls on which the decay products carry primitive charge vectors, the original techniques of [7] involving analysis of multi-centered black hole solutions are equally applicable to theories with $\mathcal{N} = 2$, 4 and 8 supersymmetries in four dimensions. In the case of $\mathcal{N} = 2$ supersymmetric theories these as well as other techniques have been developed which allow us to generalize the wall crossing formula to the cases where the final decay products carry non-primitive charge vectors [9 – 11]. However so far these techniques have not been generalized to $\mathcal{N} > 2$ supersymmetric string theories.

Recently by examining the exact formula for the dyon partition function in the special case of heterotic string theory on T^6, refs. [18, 19] studied the jump in the dyon spectrum of this theory across various walls of marginal stability, including the ones on which the decay products are non-primitive, and proposed a general wall crossing formula for such walls. The purpose of this paper is to give a macroscopic derivation of this formula from the study of multi-centered black holes in a general $\mathcal{N} = 4$ supersymmetric string theory. As we shall see, this is possible and yields results in perfect agreement with the results derived from microscopic analysis.[1] Since the jump in the index across these walls is exponentially small compared to the leading contribution, this reinforces our belief that black holes capture not only the leading contribution to the statistical entropy but also the exponentially suppressed contributions.

We begin by reviewing the derivation of the wall crossing formula in $\mathcal{N} = 2$ supersymmetric string theories. Let us consider the wall of marginal stability associated with the decay of a dyon of charge (Q, P) into a pair of primitive dyons of charges (Q_1, P_1) and (Q_2, P_2), with Q and P denoting electric and magnetic charges in some basis. In this case the jump in the index across this wall of marginal stability can be computed from the following simple argument [4 – 9]. A classical analysis shows that a two centered solution of total charge (Q, P), with one center having charges (Q_1, P_1) and the other center having

[1]The formula for the dyon partition function was originally guessed in [18, 19] by imposing various consistency conditions including known wall crossing formula for decay into primitive dyons. More recently a proof has been suggested in [21].

charges (Q_2, P_2), exist on one side of the wall and does not exist on the other side [4, 5]. Thus the jump in the index can be identified as the index associated with the two centered solution. One also finds that as we approach the wall, the separation between the two centers approaches infinity. Thus in this limit the index can be identified as the product of the index associated with each component, and the index associated with the supersymmetric quantum mechanics describing relative motion between the two centers. The latter gives a contribution of $(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,|Q_1\cdot P_2 - Q_2\cdot P_1|$. Thus if $d_h(Q_1,P_1)$ and $d_h(Q_2,P_2)$ denote the index associated with the decay products, the net change in the index will be given by

$$(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,|Q_1\cdot P_2 - Q_2\cdot P_1|\, d_h(Q_1,P_1)\, d_h(Q_2,P_2)\,. \tag{1}$$

When the charge vectors (Q_1, P_1) and/or (Q_2, P_2) are non-primitive, the above formula is known to undergo non-trivial modification [9−11].

Our goal in this paper will be to follow a similar logic for deriving the change in the index across a wall of marginal stability where a quarter BPS state in an $\mathcal{N} = 4$ supersymmetric string theory decays into a pair of half-BPS states. As we shall see, this procedure can give the wall crossing formula non only for decay into a pair of primitive dyons but also for decay into a pair of non-primitive dyons. We begin by introducing some notations. For a given charge vector (Q, P) we define helicity trace B_n via the relation [22, 23]

$$B_n(Q,P) = \frac{1}{n!} Tr_{(Q,P)}\left((-1)^F (2h)^n\right)\,, \tag{2}$$

where $Tr_{(Q,P)}$ denotes trace over states with charge (Q, P), F denotes the fermion number and h denotes the helicity of the state. Since a quarter BPS state breaks 12 out of the 16 supersymmetries, it has 12 fermion zero modes and we need at least 6 factors of $2h$ to get a non-zero answer for the supertrace. We shall denote by $d(Q, P)$ the index $-B_6(Q, P)$ associated with quarter BPS states. A typical quarter BPS supermultiplet contains states of helicities between $H - \frac{3}{2}$ and $H + \frac{3}{2}$, with the state of helicity $H + \frac{s-3}{2}$ coming with a degeneracy of $\binom{6}{s}$. It is easy to see that the contribution of these states to B_6 is given by $(-1)^{2H+1}$, with the 1/6! in the normalization factor of B_6 cancelling a factor of 6! coming from sum over s. Thus $d(Q, P)$ effectively counts the number of quarter BPS supermultiplets carrying charge (Q, P) weighted by $(-1)^{2H}$, H being the average helicity of all the states in the supermultiplet. On the other hand half-BPS states break 8 out of the 16 supersymmetries, and hence we need 4 factors of $2h$ to get a non-vanishing supertrace. We denote by $d_h(Q_i, P_i)$ the index $B_4(Q_i, P_i)$ associated with the half-BPS decay products. $d_h(Q_i, P_i)$ counts the number of half-BPS supermultiplets weighted by $(-1)^{2H}$, H being the average helicity of all the states in the half-BPS supermultiplet. Our goal is to calculate the change in $d(Q, P)$ across a wall of marginal stability in terms of $d_h(Q_i, P_i)$ associated with the half-BPS decay products.

First consider the case when both decay products are primitive.[2] In this case on one side of the wall of marginal stability we have a two centered classical black hole solution,

[2] I wish to thank F. Denef for discussion on this case.

JHEP07(2008)078

with one center carrying charge (Q_1, P_1) and the other center carrying charge (Q_2, P_2).[3] This solution ceases to exist on the other side of the wall; hence the jump in $d(Q,P)$ across this wall can be identified as the contribution to $-B_6(Q,P)$ from this two centered solution. Now since the wall crossing formula is a change in the index, we expect it to be invariant under a continuous change in the moduli.[4] This allows us to work in a region of the moduli space where one of the decay products (say with charge (Q_1, P_1)) is heavy and the other (with charge (Q_2, P_2)) is light. In this case we can describe the system as the light particle moving in the background of the heavy particle, and ignore the backreaction of the light particle on the dynamics of the heavy particle. We now note that since the heavy particle breaks 8 out of 16 supersymmetries, in order to get a non-vanishing contribution to the supertrace over the states of the heavy particle we must insert a factor of $(2h_{(1)})^4$ into the trace, $h_{(1)}$ being the helicity of the heavy particle. Since the light particle moves in the background produced by the heavy particle, it only feels 8 of the unbroken supersymmetries. Furthermore since the classical two centered solution is quarter BPS, the light particle breaks 4 out of these 8 supersymmetries. As a result it carries 4 fermion zero modes, and we must insert a factor of $(2h_{(2)})^2$ into the supertrace over the light particle degrees of freedom to get a non-vanishing answer. On the other hand since in flat space-time the light particle, being half-BPS, breaks 8 out of 16 supersymmetries, it has altogether 8 fermion zero modes in flat space-time. 4 of these zero modes must be lifted, i.e. take part in the interaction, in the presence of the heavy particle. In fact these must combine with the bosonic modes describing the physical coordinates of the light particle to describe a supersymmetric quantum mechanics with 4 supersymmetries since the final configuration has four unbroken supersymmetries. If $d_{\rm rel}$ denotes the number of supersymmetric ground states of this quantum mechanical system, weighted by $(-1)^F$, then the index $-B_6$ of the two centered dyon system will be given by

$$d_h(Q_1, P_1)\, d_h(Q_2, P_2)\, d_{\rm rel}\,. \tag{3}$$

The normalization and sign factors work out as follows. First of all we note that the coefficient of $(2h_{(1)})^4(2h_{(2)})^2$ term in $(2h_{(1)} + 2h_{(2)})^6$ is $\binom{6}{4}$. This factor of $\binom{6}{4}$ combines with the 1/6! in the definition of B_6 to give 1/4!2!. The 4! now cancels the trace of $(-1)^F(2h_{(1)})^4$ over the 8 fermion zero modes carried by the heavy state and 2! cancels the trace of $(-1)^F(2h_{(2)})^2$ over the fermion zero modes carried by the light state, leaving behind a minus sign. This minus sign compensates for the minus sign in the relation between $d(Q,P)$ and $B_6(Q,P)$.

In order to calculate $d_{\rm rel}$ we can work in a subspace of the moduli space where the original dyon with charge (Q,P) and the decay products carrying charges (Q_1, P_1) and

[3]Since both the final dyons are half-BPS states of the full $\mathcal{N}=4$ supersymmetry algebra, they are small black holes [24, 25], but this does not affect our argument.

[4]In $\mathcal{N}=2$ supersymmetric string theories the decay products are half-BPS and can themselves decay across walls of marginal stability. As a result the wall crossing formula also changes as a function of the moduli. In contrast in the $\mathcal{N}=4$ supersymmetric string theories the decay products are half-BPS states whose index does not change as we vary the moduli. Thus we expect the wall crossing formula to be unchanged as we vary the moduli.

JHEP07(2008)078

(Q_2, P_2) can be regarded as half-BPS dyons of an $\mathcal{N} = 2$ subalgebra. In this case we can examine the situation from the point of view of $\mathcal{N} = 2$ supersymmetry. In the absence of the heavy particle the light particle would break 4 of the 8 supersymmetries and hence would carry four fermion zero modes. However in the presence of the heavy particle these fermion zero modes will be lifted since the heavy particle already breaks 4 of the 8 supersymmetries and hence the light particle will not break any further supersymmetry. Thus effectively the 4 fermion zero modes of the light particle will become interacting and combine with the bosonic coordinates to give a supersymmetric quantum mechanics with four supersymmetries. This must be the same interacting quantum mechanical system that we got by analyzing the system from the $\mathcal{N} = 4$ viewpoint, — the effect of truncation to the $\mathcal{N} = 2$ subsector being simply the removal of the 4 non-interacting fermion zero modes on the light state. Now we can use the already existing results in $\mathcal{N} = 2$ theory [7] to conclude that this supersymmetric quantum mechnics has a set of supersymmetric ground states of Witten index $(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,|Q_1\cdot P_2 - Q_2\cdot P_1|$. Using (3) we now get the total jump in the index across the wall

$$\Delta d(Q,P) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,|Q_1\cdot P_2 - Q_2\cdot P_1|\, d_h(Q_1,P_1)\, d_h(Q_2,P_2)\,. \tag{4}$$

Note that the final formula is symmetric under the exchange of final state charges even though we treated them differently in our analysis.

This finishes our analysis of the wall crossing formula for decay into a pair of primitive dyons. Now consider the case where the light dyon carries a non-primitive charge vector, i.e. (Q_2, P_2) is given by some integer N_2 times a a primitive charge vector $(Q_2/N_2, P_2/N_2)$. In this case besides the two centered configuration described earlier, the system also admits multi-centered configurations, with the first center carrying charges (Q_1, P_1) and the others carrying charges $(\alpha_i Q_2/N_2, \alpha_i P_2/N_2)$, with $\alpha_i \in \mathbb{Z}$, $\alpha_i \geq 1$, $\sum_i \alpha_i = N_2$. All of these configurations cease to exist on the other side of the wall and hence could contribute to the wall crossing formula. Since we are working in a region of the moduli space where the dyons carrying charges of the form $(\alpha_i Q_2/N_2, \alpha_i P_2/N_2)$ are light, they are also weakly interacting. Thus we can regard the states of the full system as tensor products of the states of the heavy particle of charge (Q_1, P_1) and the states of light particles carrying charges $(\alpha_i Q_2/N_2, \alpha_i P_2/N_2)$ moving in the background of the heavy particle. However the correponding trace will vanish [12] since the heavy particle carries 8 fermion zero modes requiring insertion of 4 powers of helicity into the supertrace and each of the light particles carries 4 fermion zero modes, requiring insertion of 2 powers of helicity into trace over the Hilbert space of each particle. Since we only have a total of 6 powers of helicity in the definition of B_6 we cannot saturate all the fermion zero modes.

There is however an exception to this rule. The above argument assumes that the full Hilbert space is a direct product of the Hilbert spaces of the component dyons. However if some of the components are identical then we must (anti-)symmetrize the wave-function and the full Hilbert space is no longer a direct product of the Hilbert spaces of the component dyons. Such configurations could give non-vanishing contribution to the index.[5] We begin

[5]Similar issues arose in the analysis of [21].

by examining the contribution from a multi-centered dyon configurations with one heavy center of charge (Q_1, P_1) and L light centers each carrying charges $(Q_2/L, P_2/L)$, L being a factor of N_2. Furthermore we take all these L light centers in the same internal quantum state,[6] and in the same state of the supersymmetric quantum mechanics describing the motion of the light particle in the heavy particle background. We shall argue later that these are the only types of multi-centered configurations which contribute to the index.

Let us now compute the contribution to the index from such a configuration. We denote by $h_{\rm int}$ the contribution to the helicity of one of the L light dyons from the internal quantum state. This dyon also gets a contribution to the helicity from the supersymmetric quantum mechanics describing the motion of the light particle in the heavy particle background, – this is given by [7]

$$h_{\rm rel} = |Q_1 \cdot P_2 - Q_2 \cdot P_1|/(2L) - (1/2) - \text{integer}\,. \tag{5}$$

Finally quantization of the 4 fermion zero modes produces a 4-fold degenerate state, with 1 state of helicity $-1/2$, two states of helicity 0 and one state of helicity $1/2$. Let us denote these four states by $|i\rangle$ with $1 \le i \le 4$ and let $\hat{h}_i$ be the contribution to the helicity of the ith state from the fermion zero modes. Thus when we take the tensor product of a fixed internal state of the light dyon, a fixed state of the supersymmetric quantum mechanics, and the states obtained by quantizing the fermion zero mode, the states can be labelled by the index i, with total helicity

$$h_i = h_{\rm int} + h_{\rm rel} + \hat{h}_i\,. \tag{6}$$

We shall refer to these states as single particle states. Our goal is to consider an L particle state, with each of the L particles being in the same internal state and same state of the supersymmetric quantum mechanics, and calculate the contribution to B_6 from these states. For this we need to first identify the statistics of the single particle states described above. Naively the particle will be bosonic or fermionic depending on whether h_i is integer or half integer. However in the contribution to $h_{\rm rel}$ given in (5), the part proportional to $|Q_1 \cdot P_2 - Q_2 \cdot P_1|/(2L)$ comes from the angular momentum of the electromagnetic field of the dyon system and does not directly contribute to the statistics of the light particle. Thus the particle should be regarded as bosonic or fermionic depending on whether $2h_{\rm int}+2\hat{h}_i+1$ is even or odd. The L particle states are then labelled by specifying the occupation number n_i of the ith state, subject to the conditions that $\sum_i n_i = L$, n_i takes values 0 and 1 if $2h_{\rm int} + 2\hat{h}_i + 1$ is odd and n_i takes all non-negative integer values if $2h_{\rm int} + 2\hat{h}_i + 1$ is even.

Let us now denote by g_L the quantity:

$$g_L = -\frac{1}{2!} Tr_L \left((-1)^{2h} \, (2h)^2 \right) , \tag{7}$$

where Tr_L denotes trace over the Hilbert space of L particle states introduced above and h in (7) stands for the total contribution to the helicity from the L light dyons. Then the

[6] By internal quantum state of a half-BPS dyon we shall refer to supersymmetry singlet part of the state. We tensor this state with the states obtained by quantizing the fermion zero modes to get the full supermultiplet.

contribution to $-B_6$ from these states will be given by the product of g_L and $d_h(Q_1, P_1)$. Using the description of the multiparticle states given above, we get

$$g_L = -\frac{1}{2} \sum_{\{n_i\}, \sum_i n_i = L} (-1)^{2\sum_i n_i h_i} \left(2 \sum_k n_k h_k \right)^2 . \tag{8}$$

Using (6) we can write this as

$$g_L = -(-1)^{(2h_{\rm rel}-1)L}\, \widehat{g}_L \,, \tag{9}$$

where

$$\widehat{g}_L = \frac{1}{2} \sum_{\{n_i\}, \sum_i n_i = L} (-1)^{\sum_i n_i (2h_{\rm int} + 2\hat{h}_i + 1)} \left(2 \sum_k n_k (h_{\rm int} + \hat{h}_k + h_{\rm rel}) \right)^2 . \tag{10}$$

$\widehat{g}_L$ is most conveniently evaluated by first calculating the partition function

$$f(\mu, \beta) \equiv \sum_{\{n_i\}} (-1)^{\sum_i n_i (2h_{\rm int} + 2\hat{h}_i + 1)}\, e^{2\beta \sum_i n_i (h_{\rm int} + \hat{h}_i + h_{\rm rel})}\, e^{\mu \sum_i n_i} \,, \tag{11}$$

and then calculating $\widehat{g}_L$ as the coefficient of the $e^{\mu L}$ term in

$$\frac{1}{2} \left[\frac{d^2}{d\beta^2} f(\mu, \beta) \right]_{\beta=0} . \tag{12}$$

As described earlier, the sum over n_i is restricted to 0 and 1 for $2h_{\rm int} + 2\hat{h}_i + 1$ odd and to all non-negative integers for $2h_{\rm int} + 2\hat{h}_i + 1$ even. This gives

$$f(\mu, \beta) = \prod_i \left(1 - e^{\mu + 2\beta (h_{\rm int} + \hat{h}_i + h_{\rm rel})} \right)^{(-1)^{2h_{\rm int} + 2\hat{h}_i}} . \tag{13}$$

Thus

$$\begin{aligned} \ln f(\mu, \beta) &= \sum_i (-1)^{2h_{\rm int} + 2\hat{h}_i} \ln \left(1 - e^{\mu + 2\beta (h_{\rm int} + \hat{h}_i + h_{\rm rel})} \right) \\ &= -\sum_{k=1}^{\infty} \frac{1}{k} \sum_i (-1)^{2h_{\rm int} + 2\hat{h}_i}\, e^{k \left(\mu + 2\beta (h_{\rm int} + \hat{h}_i + h_{\rm rel}) \right)} \\ &= \sum_{k=1}^{\infty} \frac{1}{k} (-1)^{2h_{\rm int}}\, e^{k (\mu + 2\beta (h_{\rm int} + h_{\rm rel}))}\, (e^{k\beta/2} - e^{-k\beta/2})^2 \,, \end{aligned} \tag{14}$$

where in the second step we have expanded $\ln(1-x)$ in a Taylor series expansion in x and in the last step we have explicitly carried out the sum over i using the fact that there is one state with $\hat{h}_i = -1/2$, two states with $\hat{h}_i = 0$ and one state with $\hat{h}_i = 1/2$. This gives, from (12)

$$\hat{g}_L = (-1)^{2h_{\rm int}}\, L \,, \tag{15}$$

JHEP07(2008)078

and hence, from (5), (9),

$$g_L = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 2h_{\rm int}+1}\, L\,. \tag{16}$$

As already mentioned below (7), the net contribution to the index $-B_6$ from the specific configurations analyzed above will be given by $g_L\, d_h(Q_1, P_1)$.

The total contribution to $-B_6$ from the configurations where the light dyon is split into L identical centers is obtained by summing $g_L\, d_h(Q_1, P_1)$ over all the internal states of the dyon of charge $(Q_2/L, P_2/L)$ and all the supersymmetric ground states of the supersymmetric quantum mechanics describing the motion of a single particle of charge $(Q_2/L, P_2/L)$ in the background of the heavy particle with charge (Q_1, P_1). The former gives a factor of $d_h(Q_2/L, P_2/L)$ while the latter gives a factor of $|Q_1 \cdot P_2 - Q_2 \cdot P_1|/L$ [7]. This gives the net contribution to the index from these configurations to be

$$(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\, |Q_1 \cdot P_2 - Q_2 \cdot P_1|\, d_h(Q_1, P_1) d_h(Q_2/L, P_2/L)\,. \tag{17}$$

Note that $(-1)^{2h_{\rm int}}$ has been absorbed into the definition of $d_h(Q_2/L, P_2/L)$. Finally we must sum over all possible values of L since all multi-centered configurations with one center having charge (Q_1, P_1) and the other L centers having charges $(Q_2/L, P_2/L)$ will disappear across the wall of marginal stability on which the original dyon decays into dyons of charge (Q_1, P_1) and (Q_2, P_2). This gives the final formula for the jump in the index to be

$$(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\, |Q_1 \cdot P_2 - Q_2 \cdot P_1| \sum_{L|(Q_2,P_2)} d_h(Q_1, P_1) d_h(Q_2/L, P_2/L)\,. \tag{18}$$

One could in principle consider more general configurations where the charge vector (Q_2, P_2) splits into different groups with the members within each group being identical but members of different groups being not identical. In this case each group will require an insertion of an h^2 factor to saturate its fermion zero modes. Since a factor of h^4 is already used up by the heavy state we see that we can allow at most one group. Thus the configurations we have analyzed above are the only ones which can contribute to B_6.

This finishes our analysis of the wall crossing formula when one of the decay products is non-primitive. What about the case when both decay products are non-primitive? For this we let both (Q_1, P_1) and (Q_2, P_2) be non-primitive and continue to work in the corner of the moduli space where the dyon of charge (Q_1, P_1) is much heavier than the dyon of charge (Q_2, P_2). Now the dyon of charge (Q_1, P_1) can also split into multiple centers producing a non-spherically symmetric background for the dyons of charge $(Q_2/L, P_2/L)$. However by arguments similar to the ones given above we can conclude that unless all the centers into which (Q_1, P_1) splits are in the same quantum state, the contribution to the index from these configurations will vanish. If we use position space basis for these centers — which is the natural basis for heavy particles — we see that the different centers into which the dyon of charge (Q_1, P_1) splits must coincide in space. Thus it continues to produce a spherically symmetric potential for dyons of charge $(Q_2/L, P_2/L)$ and our previous analysis goes through except for a possible change in the overall factor associated with the index of the dyon of charge (Q_1, P_1). Thus the jump in the index must take the form:

$$(-1)^{Q_1 \cdot P_2 - Q_2 \cdot P_1 + 1} \, |Q_1 \cdot P_2 - Q_2 \cdot P_1| \sum_{L|(Q_2,P_2)} f_h(Q_1, P_1) d_h(Q_2/L, P_2/L) \,, \tag{19}$$

for some function $f_h(Q_1, P_1)$. $f_h(Q_1, P_1)$ gets a contribution of $d_h(Q_1, P_1)$ from the single dyon state but also possible additional contributions from the multi-dyon states into which the dyon of charge (Q_1, P_1) may split. To determine the form of the function f_h we can go to the corner of the moduli space where the dyon of charge (Q_2, P_2) becomes heavy and the dyon of charge (Q_1, P_1) becomes light and repeat our analysis. This gives the final form of the jump in the index for a general decay where both (Q_1, P_1) and (Q_2, P_2) are non-primitive:

$$\begin{aligned} \Delta d(Q, P) \,=\, & (-1)^{Q_1 \cdot P_2 - Q_2 \cdot P_1 + 1} \, |Q_1 \cdot P_2 - Q_2 \cdot P_1| \, \times \\ & \times \sum_{L_1|(Q_1,P_1)} d_h(Q_1/L_1, P_1/L_1) \sum_{L_2|(Q_2,P_2)} d_h(Q_2/L_2, P_2/L_2) \,. \end{aligned} \tag{20}$$

This agrees with the result of [18, 19] derived from the microscopic formula for the dyon partition function [18, 19, 21] for the special case of heterotic string theory compactified on T^6.

Acknowledgments

We would like to thank Nabamita Banerjee, Shamik Banerjee, Justin David, Frederik Denef, Dileep Jatkar and Yogesh Srivastava for useful discussions.

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **430** (1994) 485] [hep-th/9407087].

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[3] O. Bergman, *Three-pronged strings and 1/4 BPS states in $N = 4$ super-Yang-Mills theory*, *Nucl. Phys.* **B 525** (1998) 104 [hep-th/9712211].

[4] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049].

[5] F. Denef, *On the correspondence between D-branes and stationary supergravity solutions of type-II Calabi-Yau compactifications*, hep-th/0010222.

[6] F. Denef, B.R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, *JHEP* **05** (2001) 012 [hep-th/0101135].

[7] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [hep-th/0206072].

[8] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, hep-th/0304094.

[9] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, hep-th/0702146.

[10] E. Diaconescu and G.W. Moore, *Crossing the wall: branes vs. bundles*, arXiv:0706.3193.

[11] M. Kontsevich and Y. Soibelman, *Stability, wall-crossing and counting of objects in Calabi-Yau categories*, talk given by M. Kontsevich at the *XXXVIIth Paris Summer Institute "Black Holes, Black Rings and Modular Forms*, Paris France, August 13–24, 2007.

[12] F. Denef, private communication.

[13] J.R. David and A. Sen, *CHL dyons and statistical entropy function from $D1-D5$ system*, *JHEP* **11** (2006) 072 [hep-th/0605210].

[14] A. Sen, *Walls of marginal stability and dyon spectrum in $N=4$ supersymmetric string theories*, *JHEP* **05** (2007) 039 [hep-th/0702141].

[15] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023 [hep-th/0702150].

[16] A. Sen, *Two centered black holes and $N=4$ dyon spectrum*, *JHEP* **09** (2007) 045 [arXiv:0705.3874].

[17] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, *JHEP* **09** (2007) 070 [arXiv:0706.2363].

[18] S. Banerjee, A. Sen and Y.K. Srivastava, *Generalities of quarter BPS dyon partition function and dyons of torsion two*, *JHEP* **05** (2008) 101 [arXiv:0802.0544].

[19] S. Banerjee, A. Sen and Y.K. Srivastava, *Partition functions of torsion >1 dyons in heterotic string theory on T^6*, *JHEP* **05** (2008) 098 [arXiv:0802.1556].

[20] A. Sen, *$N=8$ dyon partition function and walls of marginal stability*, arXiv:0803.1014.

[21] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N=4$ string theory*, arXiv:0803.2692.

[22] A. Gregori et al., *R^2 corrections and non-perturbative dualities of $N=4$ string ground states*, *Nucl. Phys.* **B 510** (1998) 423 [hep-th/9708062].

[23] E. Kiritsis, *Introduction to non-perturbative string theory*, hep-th/9708130.

[24] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A 10** (1995) 2081 [hep-th/9504147].

[25] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [hep-th/0409148].

JHEP07(2008)078